Dynamical Processes over Networks (Diffusion)

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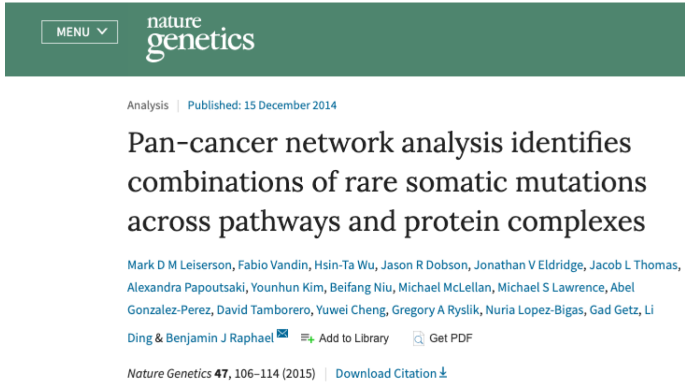
Diffusion Processes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nature genetics

Analysis | Published: 15 December 2014

Pan-cancer network analysis identifies combinations of rare somatic mutations across pathways and protein complexes

Mark D M Leiserson, Fabio Vandin, Hsin-Ta Wu, Jason R Dobson, Jonathan V Eldridge, Jacob L Thomas, Alexandra Papoutsaki, Younhun Kim, Beifang Niu, Michael McLellan, Michael S Lawrence, Abel Gonzalez-Perez, David Tamborero, Yuwei Cheng, Gregory A Ryslik, Nuria Lopez-Bigas, Gad Getz, Li Ding & Benjamin J Raphael Add to Library Get PDF

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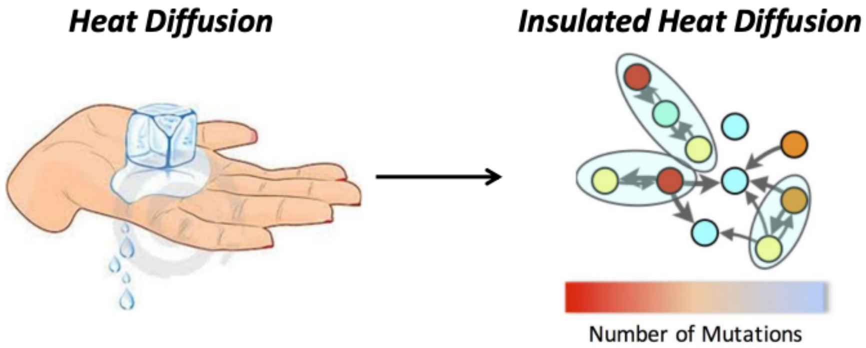

Diffusion Processes

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nature REVIEWS NEUROSCIENCE

Review Article | Published: 14 December 2017

Communication dynamics in complex brain networks

Andrea Avena-Koenigsberger, Bratislav Misic & Olaf Sporns

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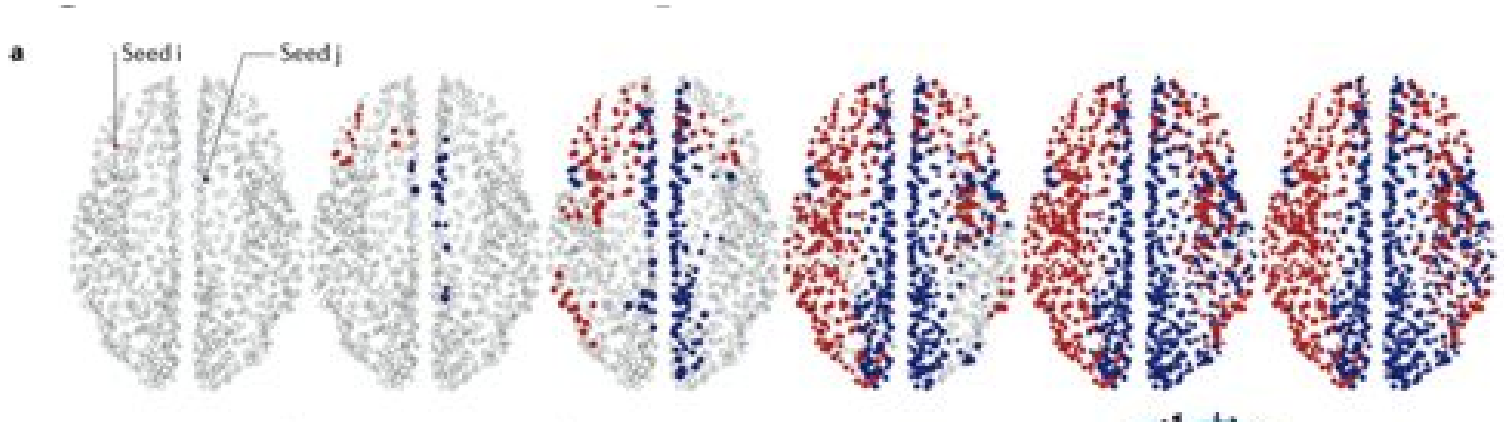

General dynamical processes

Research articles

Partitioning a reaction–diffusion ecological network for dynamic stability

Dinesh Kumar, Jatin Gupta and Soumyendu Raha

Published: 13 March 2019
https://doi-org.proxy-um.researchport.umd.edu/10.1098/rspa.2018.0524

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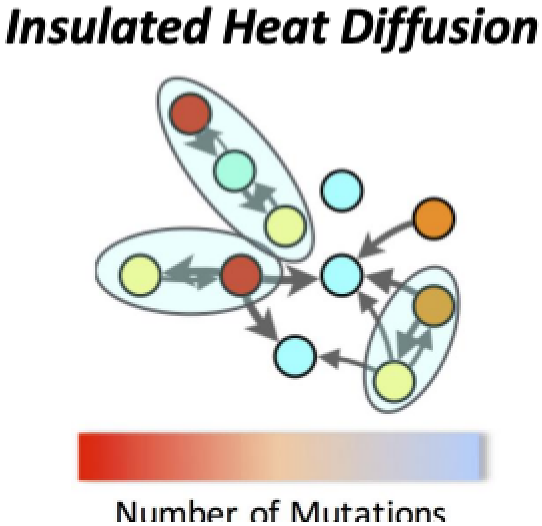
$$\left.\begin{aligned}\dot{x}_i &= f_i(x_i, y_i) + \sum_{j=1, j\neq i}^{n} w_{ij}^{x}(x_j - x_i)\\ \dot{y}_i &= g_i(x_i, y_i) + \sum_{j=1, j\neq i}^{n} w_{ij}^{y}(y_j - y_i),\end{aligned}\right\}$$

Diffusion

Data: Real-valued variable x_i for each node in network

Assumption: Network seeks stable state where x_i is "smooth" over network

Process: Nodes with high value of x_i diffuse value to neighbors with lower value x_i



Diffusion

Question: How does the value of x_i change over time?

$$\frac{dx_i}{dt} = C \sum_j A_{ij}(x_j - x_i)$$

Diffusion

In matrix form, in terms of the *Graph Laplacian*

$$\frac{d\mathbf{x}}{dt} + CL\mathbf{x} = 0$$

$$L = D - A$$

with $D = \mathrm{diag}(\mathbf{k})$, $\mathbf{k}$ the vector of node degrees.

Diffusion

Let's do the same previous trick with Eigenvalue decomposition (this time of L)

$$\mathbf{x}(t) = \sum_r a_r(t)\mathbf{v}_r$$

Diffusion

Let's do the same previous trick with Eigenvalue decomposition (this time of L)

$$\mathbf{x}(t) = \sum_r a_r(t)\mathbf{v}_r$$

Can rewrite diffusion equation as

$$\frac{da_r}{dt} + C\lambda_l a_r = 0$$

Difussion

Solution

$$a_r(t) = a_r(0)e^{-C\lambda_r t}$$

Difussion

Solution

$$a_r(t) = a_r(0)e^{-C\lambda_r t}$$

Properties

- $\lambda_r \geq 0$ for all 0 (dynamics tend to stable point)
- The smallest E.V. $\lambda_1 = 0$
- L is block-diagonal, number of 0 eigenvalues equal to number of components

General Dynamical Systems on Networks

We have now seen two examples of dynamical systems on networks

- Epidemics

$$\frac{dx_i}{dt} = \beta(1 - x_i)\sum_j A_{ij}x_j$$

- Diffusion

$$\frac{dx_i}{dt} = C\sum_j A_{ij}(x_j - x_i)$$

General Dynamical Systems on Networks

Let's look at these in the general case

$$\frac{dx_i}{dt} = f_i(x_i) + \sum_j A_{ij} g_{ij}(x_i, x_j)$$

Exercise: rewrite epidemic (SI) model and diffusion model in general framework

General Dynamical Systems on Networks

How to analyze this in the general case?

Linear stability analysis

- Stability: let's find states where dynamics are stable (attracting fixed points)
- Linear: let's simplify analysis by looking at linear approximations of dynamics around these states

Linear stability analysis

Let's forget networks for a moment. Consider dynamical system defined by

$$\frac{dx}{dt} = f(x)$$

Linear stability analysis

Let's forget networks for a moment. Consider dynamical system defined by

$$\frac{dx}{dt} = f(x)$$

Suppose there is a point x^* where $f(x^*) = 0$ (i.e., $\frac{dx^*}{dt} = 0$)

Linear stability analysis

Let's look at a point *close* to x^*; $x = x^* + \epsilon$

Linear stability analysis

Let's look at a point *close* to x^*; $x = x^* + \epsilon$

Then

$$\frac{dx}{dt} = \frac{d\epsilon}{dt} = f(x^* + \epsilon)$$

Linear stability analysis

Now we approximate it! (Taylor expansions, a.k.a. how the Iribe center can curve)

$$\frac{d\epsilon}{dt} \approx f(x^*) + \epsilon f'(x^*) = \epsilon f'(x^*)$$

Linear stability analysis

Now we approximate it! (Taylor expansions, a.k.a. how the Iribe center can curve)

$$\frac{d\epsilon}{dt} \approx f(x^*) + \epsilon f'(x^*) = \epsilon f'(x^*)$$

and solve

$$x(t) = x^* + \epsilon(0)e^{\lambda t}$$

with $\lambda = f'(x^*)$

The sign of λ gives us useful information (hold on to this thought).

Linear stability analysis

How about systems with two variables?

$$\frac{dx}{dt} = f(x, y)$$

$$\frac{dy}{dt} = g(x, y)$$

Linear stability analysis

Suppose we have fixed point (x^*, y^*)

$$f(x^*, y^*) = 0$$

$$g(x^*, y^*) = 0$$

Linear stability analysis

Under the useful condition that $\frac{df}{dy} = 0$ and $\frac{dg}{dx} = 0$ then

$$\frac{d\epsilon_x}{dt} = \lambda_1 \epsilon_x$$

$$\frac{d\epsilon_y}{dt} = \lambda_2 \epsilon_y$$

with $\lambda_1 = f'_x(x^*, y^*)$ and $\lambda_2 = g'_y(x^*, y^*)$

Linear stability analysis

And solution

$$x(t) = x^* + \epsilon_x(0)e^{\lambda_1 t}$$

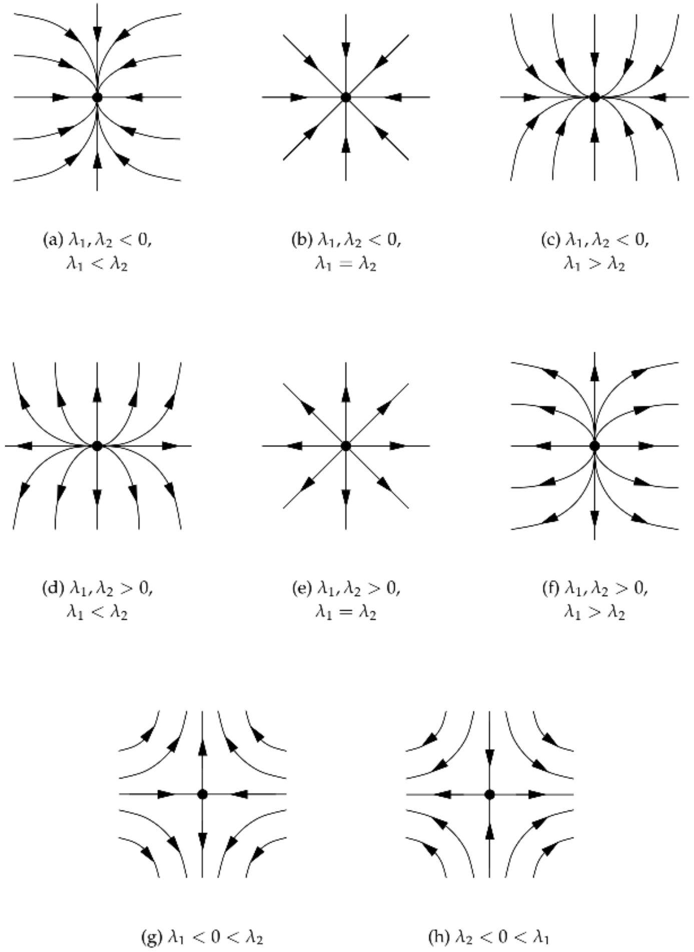
$$y(t) = y^* + \epsilon_y(0)e^{\lambda_2 t}$$

Linear stability analysis

$\lambda < 0$ *attracting* fixed point

$\lambda > 0$ *repelling* fixed point

$\lambda = 0$ ¯*(ツ)*/¯



Linear stability analysis

Back to networks

$$\frac{dx_i}{dt} = f(x_i) + \sum_j A_{ij} g(x_i, x_j)$$

Linear stability analysis

Back to networks

$$\frac{dx_i}{dt} = f(x_i) + \sum_j A_{ij} g(x_i, x_j)$$

Imagine we have a fixpoint $\{x_i^*\}$

$$f(x_i^*) = 0$$

$$g(x_i^*, x_j^*) = 0$$

Linear stability analysis

Using same approach

$$\frac{d\epsilon_i}{dt} = \left[\alpha_i \sum_j \beta_{ij} A_{ij}\right] \epsilon_i + \sum_j \gamma_{ij} A_{ij} \epsilon_j$$

with $\alpha_i = f'(x_i^*)$; $\beta_{ij} = g'_{x_i}(x_i^*, x_j^*)$; and $\gamma_{ij} = g'_{x_j}(x_i^*, x_j^*)$

Linear stability analysis

Eigenvalues again!

Letting

$$M_{ij} = \delta_{ij} \left[\alpha_i \sum_j \beta_{ij} A_{ij} \right] + \sum_j \gamma_{ij} A_{ij}$$

Linear stability analysis

Eigenvalues again!

Letting

$$M_{ij} = \delta_{ij} \left[\alpha_i \sum_j \beta_{ij} A_{ij}\right] + \sum_j \gamma_{ij} A_{ij}$$

Then

$$\frac{d\epsilon}{dt} = M\epsilon$$

Linear stability analysis

Write $\epsilon(t) = \sum_r a_r(t)\mathbf{v}_r$ where $\mathbf{v}_r$ is eigen-vector of M.

Linear stability analysis

Write $\epsilon(t) = \sum_r a_r(t)\mathbf{v}_r$ where $\mathbf{v}_r$ is eigen-vector of M.

Then solve as

$$a_r(t) = a_r(0)e^{\lambda_r t}$$

"Eigenvalues" of M determine attracting or repelling state, if at least one positive λ_r then system is not stable

Linear stability analysis

Back to Graph Laplacian

Consider case $g(x_i, x_j) = g(x_i) - g(x_j)$ (if g is identity then we can do linear diffusion as before)

and we have a symmetric fixed point $x_i^* = x^*$ for all vertices i

Linear stability analysis

Back to Graph Laplacian

Consider case $g(x_i, x_j) = g(x_i) - g(x_j)$ (if g is identity then we can do linear diffusion as before)

and we have a symmetric fixed point $x_i^* = x^*$ for all vertices i

$$\frac{d\epsilon_i}{dt} = \alpha\epsilon_i + \beta\sum_j (k_i\delta_{ij} - A_{ij})\epsilon_j$$

Linear stability analysis

Or,

$$\frac{d\epsilon}{dt} = (\alpha I + \beta L)\epsilon$$

Linear stability analysis

Or,

$$\frac{d\epsilon}{dt} = (\alpha I + \beta L)\epsilon$$

System is *stable* if $\alpha + \beta\lambda_r < 0$ for all r

Linear stability analysis

Since smallest eigenvalue of Laplacian is 0, $\alpha = f'(x^*) < 0$ is a condition for stability

Linear stability analysis

Since smallest eigenvalue of Laplacian is 0, $\alpha = f'(x^*) < 0$ is a condition for stability

Also,

$$\frac{1}{\lambda_n} > -\left(\frac{g'(x^*)}{f'(x)}\right)$$

is a condition for stability

An example

Meme network (how often does x_i share this meme)

$$f(x) = a(1 - x)$$

$$g(x_i, x_j) = \frac{bx_j}{1 + x_j} - \frac{bx_i}{1 + x_i}$$

$a > 0, b > 0$

An example

Symmetric fixed point $x_i^* = x^* = 1$

Stability conditions

(a) $\alpha = f'(x^*) < 0$?

(b) $\lambda_n < \frac{4a}{b}$

Two useful properties of Eigenvalues

- Largest eigenvalue of adj. matrix A is bounded by maximum degree $\lambda_n \geq \sqrt{k_{max}}$, so increasing degree increases largest eigenvalue and potentially lead to *unstable* system

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- Largest eigenvalue of adj. matrix A is bounded by maximum degree $\lambda_n \geq \sqrt{k_{max}}$, so increasing degree increases largest eigenvalue and potentially lead to *unstable* system

- Largest eigenvalue of Laplacian is bounded as $k_{max} \geq \lambda_n \geq 2k_{max}$, similarly, increasing maximum degree increases largest eigenvalue and potentially lead to *unstable* system

Summary

Dynamical systems can be represented over networks

Key analytical tool is linear stability analysis

- fixed points,
- linear approximation

Conditions for stability (i.e., when fixed point is strictly attracting) depend on spectral properties of graph (precisely) and maximum degree (imprecisely)